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Homotopy Operators and Identity-Based Solutions in Cubic Superstring Field Theory

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Abstract

We construct a class of nilpotent operators using the BRST current and ghost fields in superstring theory. The operator can be realized in cubic superstring field theory as a kinetic operator in the background of an identity-based solution. For a particular type of the deformed BRST operators, we find a homotopy operator and discuss its relationship to the cohomology in a similar way to the bosonic case, which has been elaborated by the authors.

1 Introduction

String field theory (SFT) has been used in exploring the non-perturbative vacuum in string theory. After the success of the Schnabl solution [1], strategy of constructing various classical solutions in SFT has been developed in terms of “ KBc subalgebra” [2] and its extension.¹ In modified cubic superstring field theory, the Erler solution [4] was constructed as a straightforward extension of the Schnabl solution. These solutions are impressive, in particular, in the sense that the vacuum energy was evaluated exactly and turned out to be equal to the D-brane tension. Furthermore, homotopy operators for the BRST operators at the solutions have been constructed as in [5], and it shows the empty cohomology at all ghost number sectors.

On the other hand, another type of classical solutions starting from [6] have been investigated, which are called a class of “identity-based solutions.” They are constructed by half-integrations of the BRST current and the c -ghost with particular functions on the identity state. Although it is difficult to evaluate its vacuum energy directly due to a singular property of the identity state,² the cohomology of the BRST operator of the theory around the solution was investigated and turned out to be empty in the ghost number one sector [10], which provides evidence that the solution represents the tachyon vacuum in bosonic SFT. Recently, a homotopy operator for the BRST operator was obtained [11] and it was applied to show moduli-independence of one loop vacuum energy at the solution.

It is natural to ask how the identity-based solutions in bosonic SFT can be extended to solutions in super SFT. Marginal solutions given in [6] were already extended to solutions in superstring field theory in [12, 13]. Here, we will extend scalar solutions given in [6] to solutions in modified cubic superstring field theory [14, 15, 16].

Actually, apart from the context of SFT, we begin with a construction of deformed BRST operators using integration of the BRST current in superstring theory with some weighting function. To find a nilpotent operator, it is necessary to introduce two other operators in terms of ghost fields. The resulting nilpotent deformed BRST operator has a similar form to that of the theory expanded around the identity-based solution in bosonic SFT. Using a method in [11], we will show that there exists a homotopy operator for a particular weighting function, which is associated with the tachyon vacuum solution in bosonic SFT.

The situation with respect to the deformed BRST operators is very similar to the bosonic SFT. We expect that it should correspond to some classical solutions in super SFT in some sense. Indeed, we will construct a class of identity-based solutions using half-integration of the BRST current in modified cubic superstring field theory. In the theory around the solution, the deformed BRST operator with the same function is reproduced. The existence of a homotopy operator shows that the solution is nontrivial although it can

¹ See [3] and references therein, for example.

² Numerical quantitative evidences have been obtained indirectly up to level 26 [7, 8, 9].

be rewritten as a pure gauge form at least formally. However, its physical interpretation is unclear so far because the obtained solution is in the theory on a BPS D-brane.

This paper is organized as follows. In the next section, we will construct a class of deformed BRST operators. In §3, we will find their homotopy operators in the case of particular functions and discuss the relationship to the conventional BRST cohomology. In §4, we will construct corresponding solutions in cubic super SFT. Finally, we will give concluding remarks in §5. Some technical details for computations are given in Appendix A and we will present a summary of the BRST cohomology in Neveu-Schwarz (NS) 0-picture and Ramond (R) $(-1/2)$ -picture in Appendix B.

2 Deformed BRST operators

We consider a deformation of the BRST operator in superstring theory using b , c and β , γ ghosts:

$$Q' = Q(f) + C(g) + \Theta(h), \quad (2.1)$$

where each operator is defined by an integration along the unit circle $|z| = 1$,

$$Q(f) = \oint \frac{dz}{2\pi i} f(z) j_B(z), \quad (2.2)$$

$$C(g) = \oint \frac{dz}{2\pi i} g(z) c(z), \quad (2.3)$$

$$\Theta(h) = \oint \frac{dz}{2\pi i} h(z) \theta(z). \quad (2.4)$$

Here, $f(z)$, $g(z)$ and $h(z)$ are some weighting functions, and $j_B(z)$ and $c(z)$ are the BRST current and the ghost field. $\theta(z)$ is defined by

$$\theta(z) = c\beta\gamma(z) - \partial c(z), \quad (2.5)$$

which is a primary field with dimension 0.

First, we will show that if we impose nilpotency on the operator Q' , the functions $g(z)$ and $h(z)$ can be related to $f(z)$, and the resulting nilpotent operator is given by

$$Q' = Q(e^\lambda) + C\left(-\frac{1}{2}(\partial\lambda)^2 e^\lambda\right) + \Theta\left(-\frac{1}{4}\partial e^\lambda\right). \quad (2.6)$$

To find the nilpotent deformed BRST operator, it is necessary to calculate the operator product expansion (OPE) among $j_B(z)$, $c(z)$ and $\theta(z)$. The BRST current is defined by

$$\begin{aligned} j_B(z) = & cT^{\text{m}}(z) + \gamma G^{\text{m}}(z) \\ & + bc\partial c(z) + \frac{1}{4}c\partial\beta\gamma(z) - \frac{3}{4}c\beta\partial\gamma(z) + \frac{3}{4}\partial c\beta\gamma(z) - b\gamma^2(z) + \frac{3}{4}\partial^2 c(z), \end{aligned} \quad (2.7)$$

where $T^m(z)$ is the energy momentum tensor and $G^m(z)$ is the supercurrent for the matter sector. Using the definition, the OPE of the BRST currents is obtained as

$$\begin{aligned}
j_B(y) j_B(z) \sim & \frac{1}{(y-z)^3} \left\{ \left(\frac{43}{8} - \frac{c^m}{2} \right) c \partial c(z) + \left(\frac{2c^m}{3} - 7 \right) \gamma^2(z) \right\} \\
& + \frac{1}{(y-z)^2} \frac{1}{2} \partial \left\{ \left(\frac{43}{8} - \frac{c^m}{2} \right) c \partial c(z) + \left(\frac{2c^m}{3} - 7 \right) \gamma^2(z) \right\} \\
& + \frac{1}{y-z} \left\{ \left(\frac{5}{4} - \frac{c^m}{12} \right) c \partial^3 c(z) + \left(\frac{c^m}{3} - 5 \right) \gamma \partial^2 \gamma(z) \right. \\
& \left. + \partial \left(\frac{1}{4} c \gamma G^m(z) + \frac{1}{2} b c \gamma^2(z) + \frac{1}{4} \beta \gamma^3(z) \right) \right\}, \tag{2.8}
\end{aligned}$$

where the matter central charge c^m is 15. Then, the OPEs of $j_B(z)$, $\theta(z)$ and $c(z)$ are calculated as

$$\begin{aligned}
j_B(y) \theta(z) \sim & \frac{1}{(y-z)^2} \left(\frac{1}{4} c \partial c(z) - \gamma^2(z) \right) \\
& + \frac{1}{y-z} \left(-c \gamma G^m(z) - 2b c \gamma^2(z) - \beta \gamma^3(z) \right), \tag{2.9}
\end{aligned}$$

$$j_B(y) c(z) \sim \frac{1}{y-z} (c \partial c(z) - \gamma^2(z)), \tag{2.10}$$

$$\theta(y) \theta(z) \sim \frac{1}{y-z} c \partial c(z). \tag{2.11}$$

If the functions $f(z)$, $g(z)$ and $h(z)$ are holomorphic in an annulus including the unit circle, anti-commutation relations among the operators (2.2), (2.3) and (2.4) can be derived from use of the OPEs (2.8), (2.9), (2.10) and (2.11):

$$\begin{aligned}
\{Q(f), Q(f)\} = & \oint \frac{dz}{2\pi i} \frac{-1}{2} (\partial f(z))^2 \left(-\frac{17}{8} c \partial c(z) + 3 \gamma^2(z) \right) \\
& + \oint \frac{dz}{2\pi i} \frac{-1}{4} \partial f^2(z) (c \gamma G^m(z) + 2b c \gamma^2(z) + \beta \gamma^3(z)), \tag{2.12}
\end{aligned}$$

$$\{Q(f), C(g)\} = \oint \frac{dz}{2\pi i} f(z) g(z) (c \partial c(z) - \gamma^2(z)), \tag{2.13}$$

$$\begin{aligned}
\{Q(f), \Theta(h)\} = & \oint \frac{dz}{2\pi i} \frac{1}{4} h(z) \partial f(z) c \partial c(z) \\
& - \oint \frac{dz}{2\pi i} h(z) \partial f(z) \gamma^2(z) \\
& - \oint \frac{dz}{2\pi i} f(z) h(z) (c \gamma G^m(z) + 2b c \gamma^2(z) + \beta \gamma^3(z)), \tag{2.14}
\end{aligned}$$

$$\{\Theta(h), \Theta(h)\} = \oint \frac{dz}{2\pi i} h^2(z) c \partial c(z). \tag{2.15}$$

The same anti-commutation relations also hold for the functions defined only on the unit circle. As in the bosonic case [6, 10], these can be obtained by Fourier analysis with the use of the commutation relations of oscillators in Appendix A.

Using (2.12), (2.13), (2.14) and (2.15), we can find the anti-commutator of the deformed BRST operator,

$$\begin{aligned}\{Q', Q'\} &= \oint \frac{dz}{2\pi i} \left\{ \frac{17}{16}(\partial f)^2 + h^2 + 2fg + \frac{1}{2}h\partial f \right\} c\partial c(z) \\ &+ \oint \frac{dz}{2\pi i} \left\{ -\frac{3}{2}(\partial f)^2 - 2fg - 2h\partial f \right\} \gamma^2(z) \\ &+ \oint \frac{dz}{2\pi i} \frac{-1}{4} \{ \partial f^2 + 8fh \} (c\gamma G^m(z) + 2bc\gamma^2(z) + \beta\gamma^3(z)).\end{aligned}\quad (2.16)$$

For nilpotency of Q' , the functions $f(z)$, $g(z)$ and $h(z)$ should satisfy the following equations:

$$\frac{17}{16}(\partial f)^2 + h^2 + 2fg + \frac{1}{2}h\partial f = 0, \quad (2.17)$$

$$-\frac{3}{2}(\partial f)^2 - 2fg - 2h\partial f = 0, \quad (2.18)$$

$$\partial f^2 + 8fh = 0. \quad (2.19)$$

From these equations, we find that the functions $g(z)$ and $h(z)$ are given in terms of $f(z)$,

$$g(z) = -\frac{(\partial f)^2}{2f}, \quad h(z) = -\frac{1}{4}\partial f(z). \quad (2.20)$$

Writing $f(z) = e^{\lambda(z)}$, we can find that the nilpotent deformed BRST operator is given by the equation (2.6).

Next, we will show that the nilpotent operator Q' can be expressed as a similarity transformation constructed by the ghost number current $j_{\text{gh}}(z) = -bc(z) - \beta\gamma(z)$. We can find the following OPEs of $j_{\text{gh}}(z)$:

$$j_{\text{gh}}(y) j_{\text{B}}(z) \sim \frac{2}{(y-z)^3} c(z) + \frac{1}{(y-z)^2} \left(\partial c(z) - \frac{1}{4}\theta(z) \right) + \frac{1}{y-z} j_{\text{B}}(z), \quad (2.21)$$

$$j_{\text{gh}}(y) c(z) \sim \frac{1}{y-z} c(z), \quad (2.22)$$

$$j_{\text{gh}}(y) \theta(z) \sim \frac{1}{y-z} \theta(z). \quad (2.23)$$

Here, we introduce an integrated operator for the ghost number current:

$$q(\lambda) = \oint \frac{dz}{2\pi i} \lambda(z) j_{\text{gh}}(z), \quad (2.24)$$

where $\lambda(z)$ is a weighting function. Similar to (2.12), (2.13), (2.14) and (2.15), we can find the following commutation-relations:

$$[q(\lambda), Q(f)] = Q(\lambda f) + C(-\partial f \partial \lambda) + \Theta \left(-\frac{1}{4} f \partial \lambda \right), \quad (2.25)$$

$$[q(\lambda), C(g)] = C(\lambda g), \quad (2.26)$$

$$[q(\lambda), \Theta(h)] = \Theta(h\lambda). \quad (2.27)$$

Let us define Q_t with a parameter t :

$$Q_t = e^{tq(\lambda)} Q_B e^{-tq(\lambda)}. \quad (2.28)$$

Using the commutation relations (2.25), (2.26) and (2.27), we can easily find that the operator Q_t is expressed as

$$Q_t = Q(f_t) + C(g_t) + \Theta(h_t). \quad (2.29)$$

where $f_t(z)$, $g_t(z)$ and $h_t(z)$ are functions with the parameter t . Differentiating both sides of (2.28) with respect to t , we have

$$\begin{aligned} \frac{d}{dt} Q_t &= Q(\lambda f_t) + C(-\partial f_t \partial \lambda) + \Theta\left(-\frac{1}{4} f_t \partial \lambda\right) \\ &\quad + C(\lambda g_t) + \Theta(h_t \lambda), \end{aligned} \quad (2.30)$$

where we have used the expression (2.29) and the commutation relations (2.25), (2.26) and (2.27). Comparing (2.30) with the derivative of (2.29), we obtain the following differential equations:

$$\frac{d}{dt} f_t = \lambda f_t, \quad (2.31)$$

$$\frac{d}{dt} g_t = -\partial f_t \partial \lambda + \lambda g_t, \quad (2.32)$$

$$\frac{d}{dt} h_t = -\frac{1}{4} f_t \partial \lambda + h_t \lambda. \quad (2.33)$$

We can easily solve these equations under the initial conditions, $f_{t=0} = 1$, $g_{t=0} = 0$, $h_{t=0} = 0$. Finally, setting $t = 1$, we find that the nilpotent BRST operator (2.6) is written as a similarity transformation:

$$Q' = e^{q(\lambda)} Q_B e^{-q(\lambda)}. \quad (2.34)$$

3 Homotopy operators in superstring theory

Here, we construct a homotopy operator for the deformed BRST operator constructed in §2 with a particular function λ .

Firstly, we note that the following OPEs:

$$j_B(y)b(z) \sim \frac{3/2}{(y-z)^3} + \frac{1}{(y-z)^2} \left(-bc(z) - \frac{3}{4} \beta \gamma(z) \right) + \frac{1}{y-z} T(z), \quad (3.1)$$

$$c(y)b(z) \sim \frac{1}{y-z}, \quad (3.2)$$

$$\theta(y)b(z) \sim \frac{1}{(y-z)^2} + \frac{1}{y-z}\beta\gamma(z), \quad (3.3)$$

which lead to following anti-commutation relations:

$$\{Q(f), b(z)\} = \frac{3}{4}\partial^2 f(z) + \partial f(z) \left(-bc(z) - \frac{3}{4}\beta\gamma(z) \right) + f(z)T(z), \quad (3.4)$$

$$\{C(g), b(z)\} = g(z), \quad (3.5)$$

$$\{\Theta(h), b(z)\} = \partial h(z) + h(z)\beta\gamma(z). \quad (3.6)$$

Hence, anti-commutator of Q' (2.6) and $b(z)$ is calculated as

$$\{Q', b(z)\} = \frac{1}{2}(\partial^2 \lambda(z))e^{\lambda(z)} + (\partial \lambda(z))e^{\lambda(z)}j_{\text{gh}}(z) + e^{\lambda(z)}T(z). \quad (3.7)$$

If we choose the function $\lambda(z)$ such that $e^{\lambda(z)}$ has a second order zero z_0 , the above relation implies that $\{Q', b(z)\}$ becomes a c-number at $z = z_0$. For example, let us take the function $\lambda = h_a^l$:

$$h_a^l(z) = \log \left(1 - \frac{a}{2}(-1)^l(z^l - (-1)^l z^{-l})^2 \right), \quad (a \geq -1/2; l = 1, 2, 3, \dots) \quad (3.8)$$

which was adopted in [6, 10] for identity-based solutions in bosonic SFT. Although $e^{h_a^l}$ does not have second order zeros in the case $a > -1/2$, $e^{h_{a=-1/2}^l}$ has second order zeros at $z_k = e^{i\frac{k-1}{l}\pi}$ for odd l and $z_k = e^{i\frac{2k-1}{2l}\pi}$ for even l ($k = 1, 2, \dots, 2l$), which are solutions to $z^{2l} + (-1)^l = 0$. Namely, the deformed BRST operator Q' with the function $\lambda = h_{a=-1/2}^l$ has a homotopy operator \hat{A} , such as

$$\{Q', \hat{A}\} = 1, \quad \hat{A}^2 = 0. \quad (3.9)$$

In this case, the operator \hat{A} , which is BPZ even and Hermitian operator, is explicitly, given by

$$\hat{A} = \sum_{k=1}^{2l} a_k l^{-2} z_k^2 b(z_k), \quad \sum_{k=1}^{2l} a_k = 1, \quad (3.10)$$

where

$$a_k = a_{l-k+2}, \quad (k = 1, 2, \dots, l+1); \quad a_k = a_{3l-k+2}, \quad (k = l+2, l+3, \dots, 2l) \quad (3.11)$$

for odd l ,

$$a_k = a_{l-k+1}, \quad (k = 1, 2, \dots, l); \quad a_k = a_{3l-k+1}, \quad (k = l+1, l+2, \dots, 2l) \quad (3.12)$$

for even l and $a_k \in \mathbb{R}$ ($k = 1, 2, \dots, 2l$). The above \hat{A} is exactly the same form as a bosonic counterpart in [11]. In the same way, using the commutation relations obtained

by differentiating (3.7), homotopy operators for the function such that e^λ has higher-order zeros investigated in [17] can be also constructed as in the bosonic case [11].

The existence of a homotopy operator \hat{A} such as (3.9) implies that the deformed BRST operator Q' has vanishing cohomology:

$$Q'\psi = 0 \quad \Leftrightarrow \quad \psi = Q'(\hat{A}\psi). \quad (3.13)$$

On the other hand, Q' can be expressed as a similarity transformation of the conventional Q_B : $Q' = e^{q(h_{-1/2}^l)} Q_B e^{-q(h_{-1/2}^l)}$ from (2.34) and, at least formally, we have

$$Q_B\Psi = 0 \quad \Leftrightarrow \quad Q'(e^{q(h_{-1/2}^l)}\Psi) = 0 \quad \Leftrightarrow \quad e^{q(h_{-1/2}^l)}\Psi = Q'(\hat{A}e^{q(h_{-1/2}^l)}\Psi). \quad (3.14)$$

Therefore, the nontrivial part of Q_B -cohomology also becomes trivial in terms of Q' -cohomology by multiplying $e^{q(h_{-1/2}^l)}$. To find out what is happening, let us more concretely see products of $e^{q(h_{-1/2}^l)}$ and the Q_B -closed states in 0-picture (§B.1) and $(-1/2)$ -picture (§B.2), which are pictures for the NS and R sector, respectively, in modified cubic superstring field theory. We decompose $q(h_{-1/2}^l)$ to positive, zero and negative mode part:

$$q(h_{-1/2}^l) = q^{(+)}(h_{-1/2}^l) + q^{(0)}(h_{-1/2}^l) + q^{(-)}(h_{-1/2}^l), \quad (3.15)$$

where

$$q^{(0)}(h_{-1/2}^l) = -q_0 \log 4, \quad q^{(\pm)}(h_{-1/2}^l) = -\sum_{n=1}^{\infty} \frac{(-1)^{n(l+1)}}{n} q_{\pm 2nl}, \quad (3.16)$$

for $j_{\text{gh}}(z) = \sum_n q_n z^{-n-1}$ and (3.8). Noting that $[q_n, q_m] = 0$ and q_0 counts the ghost number, we find

$$\begin{aligned} e^{q(h_{-1/2}^l)}\Psi &= U2^{-2}\mathcal{P}|\text{tach}\rangle_0 + U2^{-4}\mathcal{P}'\left(c_0|\text{tach}\rangle_0 + \frac{\sqrt{2}}{\sqrt{\alpha'}k^+}\gamma_{-\frac{1}{2}}|0, k_1\rangle_0\right) \\ &\quad + Q'(e^{q(h_{-1/2}^l)}\chi), \end{aligned} \quad (3.17)$$

for Q_B -closed Ψ in the NS 0-picture with $p^+ \neq 0$ from (B.6), where U is given by negative modes of j_{gh} :

$$U = \exp(q^{(-)}(h_{-1/2}^l)) = \exp\left(-\sum_{n=1}^{\infty} \frac{(-1)^{n(l+1)}}{n} q_{-2nl}\right). \quad (3.18)$$

Using the above U , we have

$$e^{q(h_{-1/2}^l)}\Psi = U2^{-1}|\mathcal{P}\rangle_{-\frac{1}{2}} + U2^{-3}(c_0 + \gamma_0\vartheta)|\mathcal{P}'\rangle_{-\frac{1}{2}} + Q'(e^{q(h_{-1/2}^l)}\chi), \quad (3.19)$$

for Q_B -closed Ψ in the R $(-1/2)$ -picture with $p^+ \neq 0$ from (B.28). Furthermore, for the sake of completeness, we also mention the exceptional case, namely zero momentum sector, as follows:

$$e^{q(h_{-1/2}^l)}\Psi = UC^{(0)}b_{-1}|\downarrow\rangle_0 + U2^{-2}\mathcal{C}_\mu^{(1)}(\alpha_{-1}^\mu + \psi_{-\frac{1}{2}}^\mu b_{-1}\gamma_{\frac{1}{2}})|\downarrow\rangle_0$$

$$\begin{aligned}
& +U2^{-4}\mathcal{C}_\mu^{(2)}\left(\alpha_{-1}^\mu c_0 + 2\psi_{-\frac{1}{2}}^\mu \gamma_{-\frac{1}{2}} + \psi_{-\frac{1}{2}}^\mu b_{-1} c_0 \gamma_{\frac{1}{2}}\right) |\downarrow\rangle_0 \\
& +U2^{-6}\mathcal{C}^{(3)}\left(2\gamma_{-\frac{1}{2}}^2 + c_{-1}c_0 + \gamma_{-\frac{1}{2}}\gamma_{\frac{1}{2}}b_{-1}c_0\right) |\downarrow\rangle_0 + Q'(e^{q(h_{-1/2}^l)}\chi), \quad (3.20)
\end{aligned}$$

for Q_B -closed Ψ in the NS 0-picture from (B.19) and

$$e^{q(h_{-1/2}^l)}\Psi = U2^{-1}A_0^a|S_a\rangle_{-\frac{1}{2}} + U2^{-3}A_1^a\gamma_0|S_a\rangle_{-\frac{1}{2}} + Q'(e^{q(h_{-1/2}^l)}\chi), \quad (3.21)$$

for Q_B -closed Ψ in the R $(-1/2)$ -picture from (B.30).

In (3.17), (3.19), (3.20) and (3.21), the same U (3.18) is multiplied on the nontrivial part of Q_B -cohomology. By multiplying the homotopy operator \hat{A} (3.10) and move it to the right, we have

$$\hat{A}U(\cdots) = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\right)U\hat{A}(\cdots) = 0, \quad (3.22)$$

where we have used $[q_n, b_m] = -b_{n+m}$, which leads to $U^{-1}b(z)U = e^{-\sum_{n=1}^{\infty}\frac{(-1)^{n(l+1)}}{n}z^{-2nl}}b(z)$. The relation (3.22) implies that, for the nontrivial part $|\varphi\rangle$ in Q_B -cohomology in the NS and R sector, all coefficients of $\hat{A}e^{q(h_{-1/2}^l)}|\varphi\rangle$ vanish in the Fock space. On the other hand, the last equation in (3.14) indicates the relation $|\varphi\rangle = Q'\hat{A}e^{q(h_{-1/2}^l)}|\varphi\rangle$, which may be interpreted that $|\varphi\rangle$ is Q' -exact outside the Fock space. Here, we regard a space in which states are expressed as a linear combination of states made of b_{-n}, c_{-m} ($n \geq 2, m \geq -1$) on the conformal vacuum $|0\rangle_{bc}$ in the bc ghost sector and β_{-r}, γ_{-s} ($r \geq P+3/2, s \geq -P-1/2$) on $|P\rangle_{\beta\gamma}$ in the $\beta\gamma$ ghost sector in P -picture as “the Fock space.” Although we should define it as a completion with respect to an appropriate norm mathematically, we leave it ambiguous at this stage. Intuitively, we could interpret that $\hat{A}e^{q(h_{-1/2}^l)}\varphi$ weakly converges to zero but not in a strong sense.

For comparison, let us comment on a similar situation on the identity-based solution with h_a^l in the bosonic SFT investigated in [11]. The nontrivial part $|\varphi\rangle$ of Q' -cohomology in terms of the Fock space found in [10] can be regarded as Q' -exact outside the Fock space by the homotopy operator \hat{A} . However, the similarity transformation from the Kato-Ogawa BRST operator: $Q' = e^{q(h_a^l)}Q_B e^{-q(h_a^l)}$ becomes ill-defined at $a = -1/2$ in the bosonic case. In fact, U_l included in $|\varphi\rangle$ [11, 10] is different from the above U (3.18).

4 Classical solutions in modified cubic superstring field theory

In §2, we have constructed a class of deformed BRST operators, which are nilpotent. In §3, we have constructed homotopy operators at the boundary of the parameter a included in the associated functions h_a^l . We expect that they may be realized as BRST operators

at appropriate classical solutions in superstring field theory. In this section, we show that it is indeed the case in the framework of modified cubic superstring field theory.

Using the analogy of a class of identity-based solutions [6] in bosonic string field theory, we expect that a string field, which is given by

$$A_c = Q_L(f)I + C_L(g)I + \Theta_L(h)I \quad (4.1)$$

with appropriate functions: f, g and h , may be a classical solution in superstring field theory. Here, we have used half integrations:

$$Q_L(f) = \int_{C_L} \frac{dz}{2\pi i} f(z) j_B(z), \quad C_L(g) = \int_{C_L} \frac{dz}{2\pi i} g(z) c(z), \quad \Theta_L(h) = \int_{C_L} \frac{dz}{2\pi i} h(z) \theta(z), \quad (4.2)$$

where the subscript C_L for each integral denotes a half unit circle corresponding to a left half of an open string. I denotes the identity state, given by the total (matter+ghost) Virasoro generators corresponding to a conformal map $2z/(1-z^2)$ in the same way as bosonic open SFT.

The string field A_c (4.1) has the ghost number one and the picture number zero in the NS sector. Therefore, it is natural to treat it in the framework of modified cubic superstring field theory, whose action is

$$S[A, \Psi] = \frac{1}{2} \langle A, Y_{-2} Q_B A \rangle + \frac{1}{3} \langle A, Y_{-2} A * A \rangle + \frac{1}{2} \langle \Psi, Y Q_B \Psi \rangle + \langle A, Y \Psi * \Psi \rangle. \quad (4.3)$$

In the above, A and Ψ are string fields in the NS and R sector, respectively. Y_{-2} and Y are inverse picture changing operators with the picture number -2 and -1 , respectively, inserted at the midpoint. $\langle \cdot, \cdot \rangle$ denotes the BPZ inner product in the small Hilbert space. (Here, we use the (β, γ) system instead of (ξ, η, ϕ) -system for the superghost sector in terms of the worldsheet CFT.) The equations of motion for the action (4.3) are

$$Y_{-2}(Q_B A + A * A) + Y \Psi * \Psi = 0, \quad Y(Q_B \Psi + A * \Psi + \Psi * A) = 0. \quad (4.4)$$

In the following, we will find the string field A_c (4.1) in the NS sector, which satisfies

$$Q_B A_c + A_c * A_c = 0, \quad (4.5)$$

by choosing appropriate functions: f, g and h . Our strategy is almost the same as the bosonic case elaborated in [6]. For a primary field σ with conformal dimension h , we have a relation from the definition of the star product of string fields in open SFT with a midpoint interaction:

$$(\Sigma_R(F) B_1) * B_2 = -(-1)^{|\sigma||B_1|} B_1 * (\Sigma_L(F) B_2), \quad (4.6)$$

where $\Sigma_L(F)$ and $\Sigma_R(F)$ are half integrations of a primary field $\sigma(z)$ with the conformal dimension h multiplied by a function $F(z)$ such as $F(-1/z) = (z^2)^{1-h} F(z)$:

$$\Sigma_L(F) = \int_{C_L} \frac{dz}{2\pi i} F(z) \sigma(z), \quad \Sigma_R(F) = \int_{C_R} \frac{dz}{2\pi i} F(z) \sigma(z), \quad (4.7)$$

and B_1, B_2 are any string fields. C_L (C_R) denotes a half unit circle with positive (negative) real part. $(-1)^{|\sigma||B_1|}$ in (4.6) is -1 in the case that both $\sigma(z)$ and B_1 are Grassmann odd and $+1$ otherwise. In the case that B_1 and/or B_2 are the identity state, which is the identity element with respect to the star product, we have

$$\Sigma_R(F)I = -\Sigma_L(F)I, \quad (4.8)$$

$$(\Sigma_L(F)I) * B = \Sigma_L(F)B, \quad B * (\Sigma_L(F)I) = -(-1)^{|\sigma||B|}\Sigma_R(F)B. \quad (4.9)$$

Assuming relations of functions:

$$f(-1/z) = f(z), \quad g(-1/z) = z^4 g(z), \quad h(-1/z) = z^2 h(z), \quad (4.10)$$

and noting that $j_B(z), c(z)$ and $\theta(z)$ are primary fields with conformal dimension 1, -1 and 0, respectively, the half integrations included in the string field (4.1) are the same form as in (4.7). Therefore, we can use the formula (4.9) and $A_c * A_c$ is computed as

$$\begin{aligned} A_c * A_c &= \left[\frac{1}{2} \{Q_L(f), Q_L(f)\} + \frac{1}{2} \{C_L(g), C_L(g)\} + \frac{1}{2} \{\Theta_L(h), \Theta_L(h)\} \right. \\ &\quad \left. + \{Q_L(f), C_L(g)\} + \{Q_L(f), \Theta_L(h)\} + \{C_L(g), \Theta_L(h)\} \right] I \\ &= \left[-\frac{7}{32} \{\Theta_L(\partial f), \Theta_L(\partial f)\} + \frac{3}{4} \{Q_B, C_L((\partial f)^2)\} + \frac{1}{4} \{Q_B, \Theta_L(f\partial f)\} \right. \\ &\quad \left. + \frac{1}{2} \{\Theta_L(h), \Theta_L(h)\} + \{Q_B, C_L(fg)\} + \{Q_B, \Theta_L(fh)\} \right. \\ &\quad \left. - \frac{3}{4} \{\Theta_L(\partial f), \Theta_L(h)\} + \{Q_B, C_L((\partial f)h)\} \right] I. \end{aligned} \quad (4.11)$$

In the second equality, we have used some relations: (A.11), (A.12) and (A.13) described in Appendix A and $f(\pm i) = 0$ is supposed in order to perform partial integrations. Using $Q_B I = 0$, (A.10) and (4.11), the left hand side of (4.5) is

$$\begin{aligned} &Q_B A_c + A_c * A_c \\ &= \left[\left\{ Q_B, C_L \left((1+f)g + \frac{3}{4}(\partial f)^2 + h\partial f \right) \right\} + \left\{ Q_B, \Theta_L \left((1+f) \left(h + \frac{1}{4}\partial f \right) \right) \right\} \right. \\ &\quad \left. - \frac{7}{32} \{\Theta_L(\partial f), \Theta_L(\partial f)\} + \frac{1}{2} \{\Theta_L(h), \Theta_L(h)\} - \frac{3}{4} \{\Theta_L(\partial f), \Theta_L(h)\} \right] I. \end{aligned} \quad (4.12)$$

Therefore, introducing a function λ , we have

$$f = e^\lambda - 1, \quad g = -\frac{1}{2}(\partial\lambda)^2 e^\lambda, \quad h = -\frac{1}{4}(\partial\lambda)e^\lambda, \quad (4.13)$$

by imposing (4.5). The function λ should satisfy

$$\lambda(-1/z) = \lambda(z), \quad \lambda(\pm i) = 0, \quad (4.14)$$

in order to guarantee the assumptions for functions (4.10) and $f(\pm i) = 0$.

Therefore, the relation (4.5) implies that the string field A_c (4.1) with (4.13) and (4.14) in the NS sector (and vanishing string field in the R sector) can be regarded as a classical solution to the equations of motion (4.4). The action around the solution is obtained by re-expanding (4.3) (and subtracting the $S[A_c, 0]$) as

$$\begin{aligned} S'[A, \Psi] &\equiv S[A + A_c, \Psi] - S[A_c, 0] \\ &= \frac{1}{2}\langle A, Y_{-2}Q'A \rangle + \frac{1}{3}\langle A, Y_{-2}A * A \rangle + \frac{1}{2}\langle \Psi, YQ'\Psi \rangle + \langle A, Y\Psi * \Psi \rangle, \end{aligned} \quad (4.15)$$

where the BRST operator Q' at the solution is defined by

$$Q'B = Q_B B + A_c * B - (-1)^{|B|} B * A_c \quad (4.16)$$

for a string field B . ($(-1)^{|B|}$ denotes the Grassmann parity of the string field B .) Substituting the concrete expression of A_c to the above formula and using the relations (4.9), the BRST operator Q' at the solution A_c is obtained:

$$\begin{aligned} Q' &= Q_B + (Q_L(f) + C_L(g) + \Theta_L(h)) + (Q_R(f) + C_R(g) + \Theta_R(h)) \\ &= Q(e^\lambda) + C \left(-\frac{1}{2}(\partial\lambda)^2 e^\lambda \right) + \Theta \left(-\frac{1}{4}(\partial\lambda)e^\lambda \right), \end{aligned} \quad (4.17)$$

where the last expression is given by integrations along a full circle. It is equal to the deformed BRST operator (2.6) investigated in §2.

Next, let us consider the relation to the expression using a similarity transformation: $Q' = e^{q(\lambda)} Q_B e^{-q(\lambda)}$ (2.34). As in the case of bosonic SFT, one expects that A_c may be related to $e^{q_L(\lambda)I} Q_B e^{-q_L(\lambda)I}$, where the symbol “ $*$ ” for the star product among string fields is omitted. For the half integrations of j_{gh} :

$$q_L(\lambda) = \int_{C_L} \frac{dz}{2\pi i} \lambda(z) j_{\text{gh}}(z), \quad q_R(\lambda) = \int_{C_R} \frac{dz}{2\pi i} \lambda(z) j_{\text{gh}}(z), \quad (4.18)$$

we note that (4.6) should be modified such as

$$(q_R(\lambda)B_1) * B_2 = -B_1 * (q_L(\lambda)B_2) + k(\lambda)B_1 * B_2, \quad k(\lambda) = \int_{C_L} \frac{dz}{2\pi i} \frac{\lambda(z)}{z}, \quad (4.19)$$

for $\lambda(-1/z) = \lambda(z)$ because the ghost number current $j_{\text{gh}} = -bc - \beta\gamma$ is not primary and is transformed as $j_{\text{gh}}(z) = w^2 j_{\text{gh}}(w) - w$ for $w = -1/z$. Then, corresponding to (4.8) and (4.9), we have the relations:

$$q_R(\lambda)I = -q_L(\lambda)I + k(\lambda)I, \quad (q_L(\lambda)I) * B = q_L(\lambda)B. \quad (4.20)$$

Using the second equation in the above and the commutation relations given in (A.14), (A.15), (A.16) and (A.17), we can calculate as follows:

$$e^{q_L(\lambda)I} Q_B e^{-q_L(\lambda)I} = (e^{q_L(\lambda)} Q_B e^{-q_L(\lambda)}) I = \left(\sum_{k=1}^{\infty} \frac{1}{k!} (\text{ad}_{q_L(\lambda)})^k Q_B \right) I$$

$$= \left(Q_L (e^\lambda - 1) - \frac{1}{2} C_L ((\partial\lambda)^2 e^\lambda) - \frac{1}{4} \Theta_L ((\partial\lambda) e^\lambda) \right) I, \quad (4.21)$$

for a function λ such as (4.14). (We have denoted as $\text{ad}_X Y \equiv [X, Y]$.) The last expression in (4.21) is equal to A_c (4.1) with (4.13). Namely, the identity-based solution A_c in the NS sector, constructed in this section, can be rewritten as a pure gauge form. Here we note that the action (4.3) is invariant under a part of gauge transformations of string fields in the NS and R sector:

$$A' = e^{-\Lambda} A e^\Lambda + e^{-\Lambda} Q_B e^\Lambda, \quad \Psi' = e^{-\Lambda} \Psi e^\Lambda, \quad (4.22)$$

with a gauge parameter string field Λ , which is Grassmann even and has ghost number zero and picture number zero.

One may think that the solution A_c is trivial from the relation (4.21). However, the action around the solution given in (4.15) is nontrivially different from the original one (4.3) for a particular type of the function λ , for example, $\lambda = h_{a=-1/2}^l$ (3.8), because the BRST operator Q' (4.17) becomes cohomologically trivial due to the existence of a homotopy operator \hat{A} (3.9) as was investigated in §3.

In the case of bosonic SFT, for $\lambda = h_{a=-1/2}^l$, the pure gauge form of the identity-based solution becomes singular [6] because the operator $e^{\pm q_L(\lambda)}$ becomes ill-defined due to the divergent factor in the normal ordered expression. It is caused by the singular OPE of the ghost number current: $j_{\text{gh}}(y)j_{\text{gh}}(z) \sim (y-z)^{-2}$. On the other hand, in the case of superstring field theory, $j_{\text{gh}}(y)j_{\text{gh}}(z)$ ($y \rightarrow z$) is regular because the ghost number current included in (4.21) is $j_{\text{gh}} = -bc - \beta\gamma$ instead of $-bc$. Therefore, the pure gauge form, which is the first expression in (4.21), is not singular even for $\lambda = h_{a=-1/2}^l$ in this sense. If we regard the pure gauge form in (4.21) as well-defined, we can interpret that the solution A_c in modified cubic superstring field theory corresponds to the solution: $\Phi_c = -q_L(\lambda)I$ in Berkovits' WZW-like superstring field theory [18] because A_c is in the small Hilbert space and the equation of motion $\eta_0(e^{-\Phi_c} Q_B e^{\Phi_c}) = 0$ is satisfied. In this case, it is also pure gauge because it can be rewritten as $\Phi_c = \eta_0(-\xi_0 q_L(\lambda)I)$ in the large Hilbert space.

5 Concluding remarks

In this paper, we have constructed a class of deformed BRST operators Q' , which is nilpotent and has ghost number one in the context of RNS formalism of open superstring. The operator Q' is given by integrations with the BRST current, c ghost, $\theta \equiv c\beta\gamma - \partial c$ and an appropriate function λ . In the case of a particular type function λ , we have found a homotopy operator for Q' such as $\{Q', \hat{A}\} = 1$, which implies that the cohomology of Q' vanishes.

In the framework of modified cubic superstring field theory, we have constructed an identity-based solution A_c in the NS sector, in the theory around which the BRST operator coincides with the deformed BRST operator, i.e., $Q' = Q_B + [A_c, \cdot]_*$. Therefore,

corresponding to a particular type of the function such as $\lambda = h_{a=-1/2}^l$, the solution seems to be nontrivial and the vanishing cohomology of Q' might imply that the D-brane vanishes as in the case of bosonic SFT.

The deformed BRST operator Q' can be re-expressed as a similarity transform of the conventional BRST operator Q_B using the integration of the ghost number current $j_{\text{gh}} = -bc - \beta\gamma$ with the function λ : $Q' = e^{q(\lambda)} Q_B e^{-q(\lambda)}$. For a particular function, such as $\lambda = h_{a=-1/2}^l$, the normal ordered form of $e^{\pm q(\lambda)}$ is ill-defined in the bosonic case, but it is not so in the superstring. Nevertheless, a homotopy operator can be found, which may imply that the corresponding identity-based solution in cubic superstring field theory, investigated in §4, would be nontrivial although it has a pure gauge expression (4.21) formally. In fact, the image of nontrivial part $|\varphi\rangle$ of the Q_B -cohomology: $e^{q(h_{-1/2}^l)}|\varphi\rangle$ is mapped to the states outside the Fock space by the homotopy operator \hat{A} in the sense that all coefficients of $\chi' \equiv \hat{A}e^{q(h_{-1/2}^l)}|\varphi\rangle$ vanish in the Fock space and $e^{q(h_{-1/2}^l)}|\varphi\rangle = Q'\chi'$ holds as far as we respect the relation $\{Q', \hat{A}\} = 1$. It is necessary to define space of states or string fields more rigorously in order to clarify this delicate issue.

The identity-based solution, which we have constructed, is a universal solution in the cubic superstring field theory. The solution might represent a vacuum where a D-brane vanishes because the cohomology of the BRST operator becomes empty. However, this interpretation might be obscure because the original theory is on a BPS D-brane, which should be stable. The situation may be similar to the Erler solution [4], at which there exists a homotopy operator and its energy reproduces a D-brane tension.

To confirm the non-triviality of the obtained solution for particular functions, such as $\lambda = h_{a=-1/2}^l$, the evaluations of the vacuum energy or the gauge invariant overlap³ are desired. However, direct computations are difficult as in the case of bosonic SFT because A_c is an identity-based solution. Namely, indefinite quantities will appear because the identity state has vanishing width in terms of the sliver frame. To avoid this singularity, we can map the solution of $Q_B A + A * A = 0$ to other solution, which is given by a linear combination of states with finite width, as was elaborated in [21], for example. In principle, we can evaluate the vacuum energy or the gauge invariant overlap for the mapped solution as a regularization. In the action of modified cubic superstring field theory, the inverse picture changing operators, Y and Y_{-2} , are inserted at the midpoint, which might cause another problem. However, we find that the OPEs: $j_B(y)Y(z)$, $c(y)Y(z)$, $\theta(y)Y(z)$ ($y \rightarrow z$) are regular, where $Y = Y(i) = c(i)\delta'(\gamma(i))$ and $Y_{-2} = Y(i)Y(-i)$ in terms of the (β, γ) system. Therefore, there are no apparent divergences caused by the collisions of Y, Y_{-2} and the operators on the identity state in the solution A_c .

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³ See [19, 20] for details.

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A Commutation relations

In this appendix, we collect some formulae related to (anti-)commutation relations.

We expand $j_B, c, \theta, j_{\text{gh}}$ as $j_B(z) = \sum_n Q_n z^{-n-1}, c(z) = \sum_n c_n z^{-n+1}, \theta(z) = \sum_n \theta_n z^{-n}$ as usual. The anti-commutation relations among them can be derived from their OPEs, (2.8), (2.9), (2.10) and (2.11):

$$\begin{aligned} \{Q_n, Q_m\} &= -\frac{nm}{2} \oint \frac{dz}{2\pi i} z^{n+m-2} \left(-\frac{17}{8} c \partial c(z) + 3\gamma^2(z) \right) \\ &\quad + \frac{n+m}{4} \oint \frac{dz}{2\pi i} z^{n+m-1} \left(-c\gamma G^{\text{m}}(z) - 2bc\gamma^2(z) - \beta\gamma^3(z) \right), \end{aligned} \quad (\text{A.1})$$

$$\{Q_n, c_m\} = \oint \frac{dz}{2\pi i} z^{n+m-2} (c \partial c(z) - \gamma^2(z)), \quad (\text{A.2})$$

$$\begin{aligned} \{Q_n, \theta_m\} &= n \oint \frac{dz}{2\pi i} z^{n+m-2} \left(\frac{1}{4} c \partial c(z) - \gamma^2(z) \right) \\ &\quad + \oint \frac{dz}{2\pi i} z^{n+m-1} \left(-c\gamma G^{\text{m}}(z) - 2bc\gamma^2(z) - \beta\gamma^3(z) \right), \end{aligned} \quad (\text{A.3})$$

$$\{\theta_n, \theta_m\} = \oint \frac{dz}{2\pi i} z^{n+m-2} c \partial c(z), \quad (\text{A.4})$$

which imply following relations

$$\{Q_n, Q_m\} = nm \left(-\frac{7}{16} \{\theta_n, \theta_m\} + \frac{3}{2} \{Q_B, c_{n+m}\} \right) + \frac{n+m}{4} \{Q_B, \theta_{n+m}\}, \quad (\text{A.5})$$

$$\{Q_n, c_m\} = \{Q_B, c_{n+m}\}, \quad (\text{A.6})$$

$$\{Q_n, \theta_m\} = \{Q_B, \theta_{n+m}\} + n \left(-\frac{3}{4} \{\theta_n, \theta_m\} + \{Q_B, c_{n+m}\} \right). \quad (\text{A.7})$$

Similarly, by expanding the ghost number current as $j_{\text{gh}}(z) = \sum_n q_n z^{-n-1}$, we find the commutation relations:

$$[q_n, Q_m] = -mnc_{n+m} + Q_{n+m} - \frac{n}{4} \theta_{n+m}, \quad [q_n, c_m] = c_{n+m}, \quad [q_n, \theta_m] = \theta_{n+m}, \quad (\text{A.8})$$

from the OPEs given in (2.21), (2.22) and (2.23). In the context of a construction of identity-based solutions in superstring field theory, it is necessary to compute (anti-)commutation relations including half integrations: (4.2) and (4.18). Using the above

relations: (A.5), (A.6), (A.7) and (A.8), and the splitting properties of the delta function, $\delta(z, w) = \sum_n z^{n-1} w^{-n}$, clarified in [6]:

$$\int_{C_L} \frac{dz}{2\pi i} \int_{C_L} \frac{dw}{2\pi i} F(z) G(w) \delta(z, w) = \int_{C_L} \frac{dz}{2\pi i} F(z) G(z), \quad (\text{A.9})$$

we have

$$\{Q_B, Q_L(f)\} = \frac{1}{4} \{Q_B, \Theta_L(\partial f)\}, \quad (\text{A.10})$$

$$\begin{aligned} \{Q_L(f), Q_L(f)\} &= -\frac{7}{16} \{\Theta_L(\partial f), \Theta_L(\partial f)\} + \frac{3}{2} \{Q_B, C_L((\partial f)^2)\} \\ &\quad + \frac{1}{2} \{Q_B, \Theta_L(f \partial f)\}, \end{aligned} \quad (\text{A.11})$$

$$\{Q_L(f), C_L(g)\} = \{Q_B, C_L(fg)\}, \quad (\text{A.12})$$

$$\{Q_L(f), \Theta_L(h)\} = \{Q_B, \Theta_L(fh)\} - \frac{3}{4} \{\Theta_L(\partial f), \Theta_L(h)\} + \{Q_B, C_L((\partial f)h)\}, \quad (\text{A.13})$$

$$[q_L(\lambda), Q_B] = -\frac{1}{4} \Theta_L(\partial \lambda) + Q_L(\lambda), \quad (\text{A.14})$$

$$[q_L(\lambda), Q_L(\Lambda)] = -C_L(\partial \lambda \partial \Lambda) - \frac{1}{4} \Theta_L((\partial \lambda) \Lambda) + Q_L(\lambda \Lambda), \quad (\text{A.15})$$

$$[q_L(\lambda), C_L(\Lambda)] = C_L(\lambda \Lambda), \quad (\text{A.16})$$

$$[q_L(\lambda), \Theta_L(\Lambda)] = \Theta_L(\lambda \Lambda). \quad (\text{A.17})$$

Here, functions: f, λ and Λ should vanish at the points: $z = \pm i$, which are the boundaries of the half unit circle, in the above formulae including $\partial f, \partial \lambda, \partial \Lambda$ to perform partial integrations.

B On the BRST cohomology for RNS string

In this appendix, we summarize the results of the conventional Q_B -cohomology in the Fock space without b_0 (and β_0) gauge condition in 0-picture and $(-1/2)$ -picture with a brief outline of proofs. In §3, we discuss the relation between nontrivial part of the Q_B -cohomology and the homotopy operator for Q' . Here, we note that

$$\beta_r |P\rangle_{\beta\gamma} = 0 \quad (r > -P - \frac{3}{2}), \quad \gamma_r |P\rangle_{\beta\gamma} = 0 \quad (r > P + \frac{1}{2}), \quad (\text{B.1})$$

in the $\beta\gamma$ -sector in P -picture.

B.1 Neveu-Schwarz 0-picture

Let us review the Q_B -cohomology in the NS sector in terms of 0-picture. When we restrict the Fock space to $\text{Ker } b_0$ and $p^+ \neq 0$, nontrivial part of the cohomology can be written using the DDF operators in the matter sector. Namely, we have [22]

$$Q_B \Psi = 0, \quad b_0 \Psi = 0 \quad \Leftrightarrow \quad \Psi = \mathcal{P} |\text{tach}\rangle_0 + Q_B \chi, \quad (\text{B.2})$$

where $\chi \in \text{Ker } b_0$, \mathcal{P} is made of the DDF operators, which (anti-)commute with Q_B , and $|\text{tach}\rangle_0$ is the onshell tachyon in the 0-picture:

$$|\text{tach}\rangle_0 = \left(\psi_{-\frac{1}{2}}^- - \frac{1}{\sqrt{2\alpha'}k^+} b_{-1} \gamma_{\frac{1}{2}} + \frac{1}{4\alpha'(k^+)^2} \psi_{-\frac{1}{2}}^+ \right) |0, k_1\rangle_0, \quad (\text{B.3})$$

$$|0, k_1\rangle_0 = |k_1\rangle_{\text{mat}} \otimes c_1 |0\rangle_{bc} \otimes |P=0\rangle_{\beta\gamma}, \quad k_1^+ = k^+, \quad k_1^- = \frac{1}{4\alpha'k^+}, \quad k^i = 0. \quad (\text{B.4})$$

Using this result,⁴ we can demonstrate

$$Q_B \Psi = 0 \Leftrightarrow \Psi = \mathcal{P} |\text{tach}\rangle_0 + \mathcal{P}' \left(c_0 |\text{tach}\rangle_0 + \frac{\sqrt{2}}{\sqrt{\alpha'}k^+} \gamma_{-\frac{1}{2}} |0, k_1\rangle_0 \right) + Q_B \chi, \quad (\text{B.6})$$

for the Fock space without b_0 -gauge condition. Here, \mathcal{P} and \mathcal{P}' are generated by the DDF operators.

To derive (B.6), we note the similarity transformed expression of the BRST operator obtained in [25]:

$$Q_B = e^{-R_2 - R_3} (c_0 L + \sqrt{2\alpha'} A) e^{R_2 + R_3}, \quad (\text{B.7})$$

where $L = \{b_0, Q_B\}$,

$$A = p^+ \left(\sum_{n \neq 0} c_{-n} \alpha_n^- + \sum_{r \in \mathbb{Z} + \frac{1}{2}} \gamma_{-r} \psi_r^- \right), \quad (\text{B.8})$$

and

$$\begin{aligned} R_2 = & \frac{1}{\sqrt{2\alpha'} p^+} \left[\sum_{\substack{m \neq 0, n \neq 0, \\ m+n \neq 0}} \left(\frac{m+n}{2mn} \alpha_{-m}^+ \alpha_{-n}^+ \alpha_{m+n}^- + \frac{n}{m} \alpha_{-m}^+ c_{-n} b_{m+n} \right) \right. \\ & + \sum_{\substack{m, n, \\ m \neq 0}} \frac{1}{2m} \alpha_{-m}^+ \alpha_{-n}^i \alpha_{m+n}^i + \sum_{\substack{n \in \mathbb{Z}, \\ r \in \mathbb{Z} + \frac{1}{2}}} \psi_{-r}^+ \alpha_{-n}^i \psi_{n+r}^i \\ & \left. + \sum_{\substack{n \neq 0, \\ r \in \mathbb{Z} + \frac{1}{2}}} \left\{ \left(\frac{3}{2} + \frac{r}{n} \right) \alpha_{-n}^+ \psi_{-r}^+ \psi_{n+r}^- + \left(\frac{1}{4} + \frac{r}{2n} \right) \alpha_{-n}^+ \psi_{-r}^i \psi_{n+r}^i \right\} \right] \end{aligned}$$

⁴ We can apply a similar method to [23] for bosonic string, with a slight modification caused by $M|\text{tach}\rangle_0 \neq 0$. In the case of (-1) -picture, where the physical tachyon $|0, k_1\rangle_{-1} = \delta(\gamma_{\frac{1}{2}})|0, k_1\rangle_0$ satisfies $M|0, k_1\rangle_{-1} = 0$ as in bosonic string, the result is [24]:

$$Q_B \Psi = 0 \Leftrightarrow \Psi = \mathcal{P} |0, k_1\rangle_{-1} + c_0 \mathcal{P}' |0, k_1\rangle_{-1} + Q_B \chi. \quad (\text{B.5})$$

Multiplying the picture changing operator $X(z) = \{Q_B, \Theta(\beta(z))\} = G(z)\delta(\beta(z)) - \partial b(z)\delta'(\beta(z))$ and taking the limit $z \rightarrow 0$, we can also rederive the relation (B.6) in 0-picture.

$$- \left(\frac{1}{2} + \frac{r}{n} \right) \alpha_{-n}^+ \gamma_{-r} \beta_{n+r} + n \psi_{-r}^+ c_{-n} \beta_{n+r} - \psi_{-r}^+ b_{-n} \gamma_{n+r} \Bigg\} \Bigg], \quad (\text{B.9})$$

$$R_3 = \frac{1}{\sqrt{2\alpha'} p^+} b_0 \left(\sum_{n \neq 0} c_{-n} \alpha_n^+ + \sum_{r \in \mathbb{Z} + \frac{1}{2}} \gamma_{-r} \psi_r^+ \right). \quad (\text{B.10})$$

Then, we find the following relations:

$$e^{-R_2 - R_3} c_0 e^{R_2 + R_3} = c_0 - [R_3, c_0], \quad (\text{B.11})$$

$$Q_B(c_0 - [R_3, c_0]) |\text{tach}\rangle_0 = 0, \quad (\text{B.12})$$

$$-[R_3, c_0] |\text{tach}\rangle_0 = \frac{\sqrt{2}}{\sqrt{\alpha'} k^+} \gamma_{-\frac{1}{2}} |0, k_1\rangle_0 + Q_B \left(\frac{-1}{2\alpha' (k^+)^2} \psi_{-\frac{1}{2}}^+ |0, k_1\rangle_0 \right). \quad (\text{B.13})$$

Firstly, we apply (B.2) for the sector projected by $c_0 b_0$ and then, using (B.2), (B.12) and (B.13) for $\text{Ker } b_0$, we can conclude (B.6).

In the case of zero momentum states, we cannot apply (B.6) due to $p^+ = 0$. However, for $L \neq 0$ sector, the Q_B -cohomology is trivial because of $\{Q_B, b_0/L\} = 1$. Hence, we investigate the Q_B -cohomology in $\text{Ker } L$. Furthermore, we consider cohomology for each ghost number sector, where its dimension is finite although there is a creation operator $\gamma_{\frac{1}{2}}$ with negative level: $-\frac{1}{2}$, in 0-picture unlike the case of (-1) -picture.⁵ For $\Psi = |\psi\rangle + c_0 |\phi\rangle$ ($|\psi\rangle, |\phi\rangle \in \text{Ker } L \cap \text{Ker } b_0$), $Q_B \Psi = 0$ imposes $\tilde{Q} |\phi\rangle = 0$, where

$$Q_B = c_0 L + b_0 M + \tilde{Q}. \quad (\text{B.15})$$

Furthermore, \tilde{Q} , which does not include b_0 and c_0 , can be expanded with respect to $\gamma_{\frac{1}{2}}$ and its canonical conjugate $\beta_{-\frac{1}{2}}$ as

$$\tilde{Q} = \gamma_{\frac{1}{2}} \hat{G}_{-\frac{1}{2}} - b_{-1} \gamma_{\frac{1}{2}}^2 + \hat{K}_{\frac{1}{2}} \beta_{-\frac{1}{2}} + \hat{Q}. \quad (\text{B.16})$$

Noting that there exists a non-negative integer N for a finite ghost number sector, $|\phi\rangle$ can be expanded as

$$|\phi\rangle = \sum_{k=0}^N \gamma_{\frac{1}{2}}^k |\phi_k\rangle, \quad \beta_{-\frac{1}{2}} |\phi_k\rangle = 0, \quad (\text{B.17})$$

⁵ In (-1) -picture, the Q_B -cohomology can be easily investigated by considering all ghost number sector in $\text{Ker } L$ with zero momentum. The result is [24]:

$$Q_B \Psi = 0 \Leftrightarrow \Psi = \mathcal{C}^{(0)} \beta_{-\frac{1}{2}} |\downarrow\rangle_{-1} + \mathcal{C}_\mu^{(1)} \psi_{-\frac{1}{2}}^\mu |\downarrow\rangle_{-1} + \mathcal{C}_\mu^{(2)} \psi_{-\frac{1}{2}}^\mu c_0 |\downarrow\rangle_{-1} + \mathcal{C}^{(3)} \gamma_{-\frac{1}{2}} c_0 |\downarrow\rangle_{-1} + Q_B \chi, \quad (\text{B.14})$$

where $\mathcal{C}^{(0)}, \mathcal{C}_\mu^{(1)}, \mathcal{C}_\mu^{(2)}$ and $\mathcal{C}^{(3)}$ are constants and $|\downarrow\rangle_{-1} = |0\rangle_{\text{mat}} \otimes c_1 |0\rangle_{bc} \otimes |P = -1\rangle_{\beta\gamma}$. By multiplying the picture changing operator $X(z)$ to (B.14) and taking the limit $z \rightarrow 0$, we can also rederive (B.19) by subtracting Q_B -exact states.

and the condition $\tilde{Q}|\phi\rangle = 0$ imposes $b_{-1}|\phi_N\rangle = 0$, which implies $|\phi_N\rangle = b_{-1}|\phi'_N\rangle$ ($\exists |\phi'_N\rangle \in \text{Ker } b_0 \cap \text{Ker } \beta_{-\frac{1}{2}}$). Then, $|\phi\rangle + Q_B(\gamma_{\frac{1}{2}}^{N-2}|\phi'_N\rangle)$ cancels the $O(\gamma_{\frac{1}{2}}^N)$ -term for $N-2 \geq 0$. Repeating this procedure, $|\phi\rangle$ can be rewritten as

$$|\phi\rangle = |\phi'_0\rangle + \gamma_{\frac{1}{2}}|\phi'_1\rangle + Q_B\chi_\phi, \quad |\phi'_0\rangle, |\phi'_1\rangle \in \text{Ker } b_0 \cap \text{Ker } \beta_{-\frac{1}{2}}, \quad \chi_\phi \in \text{Ker } b_0, \quad (\text{B.18})$$

with $L|\phi'_0\rangle = 0$ and $L|\phi'_1\rangle = \frac{1}{2}|\phi'_1\rangle$. Using the above also for $|\psi\rangle$, we find all solutions to $Q_B\Psi = 0$ up to Q_B -exact terms. The result for zero momentum sector is

$$\begin{aligned} Q_B\Psi = 0 & \Leftrightarrow \\ \Psi = \mathcal{C}^{(0)}b_{-1}|\downarrow\rangle_0 + \mathcal{C}_\mu^{(1)}(\alpha_{-1}^\mu + \psi_{-\frac{1}{2}}^\mu b_{-1}\gamma_{\frac{1}{2}})|\downarrow\rangle_0 + \mathcal{C}_\mu^{(2)}\left(\alpha_{-1}^\mu c_0 + 2\psi_{-\frac{1}{2}}^\mu \gamma_{-\frac{1}{2}} + \psi_{-\frac{1}{2}}^\mu b_{-1}c_0\gamma_{\frac{1}{2}}\right)|\downarrow\rangle_0 \\ & + \mathcal{C}^{(3)}\left(2\gamma_{-\frac{1}{2}}^2 + c_{-1}c_0 + \gamma_{-\frac{1}{2}}\gamma_{\frac{1}{2}}b_{-1}c_0\right)|\downarrow\rangle_0 + Q_B\chi, \end{aligned} \quad (\text{B.19})$$

where $\mathcal{C}^{(0)}, \mathcal{C}_\mu^{(1)}, \mathcal{C}_\mu^{(2)}$ and $\mathcal{C}^{(3)}$ are constants and $|\downarrow\rangle_0 = |0\rangle_{\text{mat}} \otimes c_1|0\rangle_{bc} \otimes |P=0\rangle_{\beta\gamma}$.

B.2 Ramond $(-1/2)$ -picture

Here, we review the Q_B -cohomology in the R sector with $(-1/2)$ -picture [26, 24]. Firstly, we note that the $L(\equiv \{Q_B, b_0\}) \neq 0$ sector is Q_B -trivial also in the R sector. By restricting the Fock space to $\text{Ker } b_0 \cap \text{Ker } \beta_0$ and $p^+ \neq 0$, the Q_B -cohomology was investigated in [27] and the result is

$$Q_B\Psi = 0, \quad b_0\Psi = 0, \quad \beta_0\Psi = 0 \Leftrightarrow \Psi = |\mathcal{P}\rangle_{-\frac{1}{2}} + Q_B\chi, \quad (\text{B.20})$$

where $|\mathcal{P}\rangle_{-\frac{1}{2}}$ indicates the states generated by the transverse operators, which (anti-)commute with Q_B , in the matter sector and $c_1|0\rangle_{bc} \otimes |P=-1/2\rangle_{\beta\gamma}$ in the ghost sector. Let us remove the b_0 and β_0 gauge condition. For $\Psi = |\psi\rangle + c_0|\phi\rangle$ ($|\psi\rangle, |\phi\rangle \in \text{Ker } L \cap \text{Ker } b_0$), $Q_B\Psi = 0$ imposes $\tilde{Q}|\phi\rangle = 0$, where \tilde{Q} is also defined by $Q_B = c_0L + b_0M + \tilde{Q}$ in the R sector. Furthermore, \tilde{Q} , which does not include b_0 and c_0 , can be expanded with respect to γ_0 and its canonical conjugate β_0 as

$$\tilde{Q} = \gamma_0 F + \beta_0 K + \hat{Q}, \quad (\text{B.21})$$

$$F = \sum_{n \in \mathbb{Z}} \alpha_{-m} \cdot \psi_m + \sum_{n \neq 0} \left(\frac{n}{2} \beta_n c_{-n} - 2b_{-n} \gamma_n \right). \quad (\text{B.22})$$

We suppose that there exists a non-negative integer N such that

$$|\phi\rangle = \sum_{k=0}^N \gamma_0^k |\phi_k\rangle, \quad \beta_0 |\phi_k\rangle = 0. \quad (\text{B.23})$$

Then, the condition $\tilde{Q}|\phi\rangle = 0$ imposes $F|\phi_N\rangle = 0$, which implies $|\phi_N\rangle = F|\phi'_N\rangle$ ($\exists |\phi'_N\rangle \in \text{Ker } b_0 \cap \text{Ker } \beta_0$), because $F^2 = L$ (i.e., F : nilpotent in $\text{Ker } L$) and $\{F, \vartheta\} = 1$ with

$\vartheta \equiv \psi_0^+ / (\sqrt{2\alpha'} p^+)$. Therefore, $|\phi\rangle - Q_B(\gamma_0^{N-1}|\phi'_N\rangle)$ cancels the $O(\gamma_0^N)$ -term for $N-1 \geq 0$. Repeating this procedure, $|\phi\rangle$ can be rewritten as

$$|\phi\rangle = |\phi'_0\rangle + Q_B \chi_\phi, \quad |\phi'_0\rangle \in \text{Ker } L \cap \text{Ker } b_0 \cap \text{Ker } \beta_0, \quad \chi_\phi \in \text{Ker } b_0. \quad (\text{B.24})$$

Using this fact and (B.20), we have

$$Q_B \Psi = 0 \quad \Rightarrow \quad |\Psi\rangle = |\psi\rangle + c_0 |\mathcal{P}\rangle_{-\frac{1}{2}} + Q_B \chi_\phi, \quad (\text{B.25})$$

by redefining $|\psi\rangle, \chi_\phi \in \text{Ker } b_0$ appropriately, where $|\mathcal{P}\rangle_{-\frac{1}{2}}$ denotes the transverse states. As in the case of NS 0-picture in §B.1, we should note that $M|\mathcal{P}\rangle_{-\frac{1}{2}} \neq 0$ and the similarity transformed expression of Q_B (B.7) holds for A, R_2 and R_3 by replacing $\sum_{r \in \mathbb{Z} + \frac{1}{2}}$ with $\sum_{r \in \mathbb{Z}}$ in (B.8), (B.9) and (B.10). Then, we find the relations: (B.11) and

$$Q_B(c_0 - [R_3, c_0])|\mathcal{P}\rangle_{-\frac{1}{2}} = 0, \quad (\text{B.26})$$

$$-[R_3, c_0]|\mathcal{P}\rangle_{-\frac{1}{2}} = \gamma_0 \vartheta |\mathcal{P}\rangle_{-\frac{1}{2}}. \quad (\text{B.27})$$

Using the above, we can show

$$Q_B \Psi = 0 \quad \Leftrightarrow \quad \Psi = |\mathcal{P}\rangle_{-\frac{1}{2}} + (c_0 + \gamma_0 \vartheta) |\mathcal{P}'\rangle_{-\frac{1}{2}} + Q_B \chi, \quad (\text{B.28})$$

where $|\mathcal{P}\rangle_{-\frac{1}{2}}$ and $|\mathcal{P}'\rangle_{-\frac{1}{2}}$ are transverse states in $\text{Ker } L \cap \text{Ker } b_0 \cap \text{Ker } \beta_0$.

In the zero momentum sector, (B.28) cannot be applied because $\vartheta = \psi_0^+ / (\sqrt{2\alpha'} p^+)$ is not well-defined due to $p^+ = 0$. However, noting that M includes $-\gamma_0^2$ and $\text{Ker } L$ is spanned by states of the form:

$$\Psi = \sum_{k=0}^{\infty} A_k^a \gamma_0^k |S_a\rangle_{-\frac{1}{2}} + c_0 \sum_{k=0}^{\infty} B_k^a \gamma_0^k |S_a\rangle_{-\frac{1}{2}}, \quad (\text{B.29})$$

where $|S_a\rangle_{-\frac{1}{2}}$ is a ground state with space-time spinor index a and A_k^a, B_k^a are constants, we can demonstrate [24]:

$$Q_B \Psi = 0 \quad \Leftrightarrow \quad \Psi = A_0^a |S_a\rangle_{-\frac{1}{2}} + A_1^a \gamma_0 |S_a\rangle_{-\frac{1}{2}} + Q_B \chi, \quad (\text{B.30})$$

in the zero momentum sector in $(-1/2)$ -picture.

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